Mean-variance and hierarchical risk parity: An empirical study of large-cap stock portfolios

Guanqi Zhu, MAFN masters degree candidate, Department of Mathematics, Columbia University.

Jonathan Schachter, Ph.D., CEO, Delta Vega Inc.

Abstract: This study compares two portfolio strategies: Markowitz Mean Variance (MV) and Hierarchical Risk Parity (HRP). We investigate how HRP's robust, diversification-focused approach performs against MV's theoretically optimal but often unstable method, using real-world data from 15 large-cap stocks.

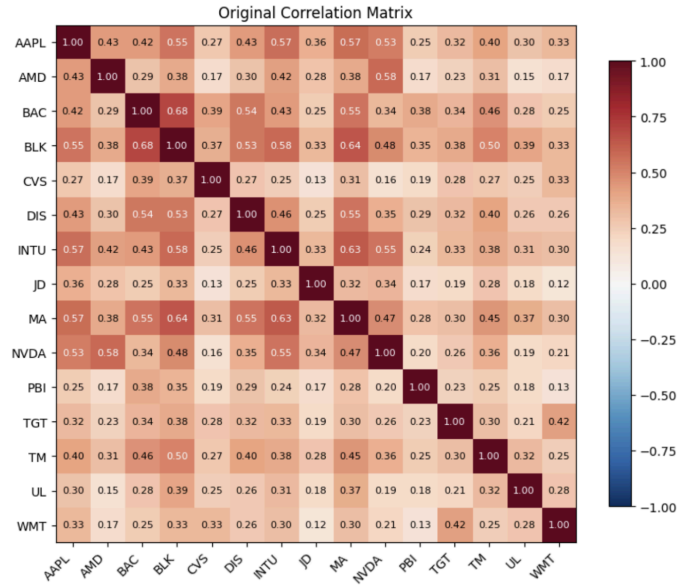
Introduction & Problem

- **Markowitz (MV) Model**: As the cornerstone of modern portfolio theory, the MV model aims to maximize risk-adjusted returns by balancing expected return and variance. However, in practice, it faces two major challenges:
    - **Data Instability**: The model relies on precise estimates of future returns and a stable covariance matrix, which are often uncertain and noisy.
    - **Matrix Pathology**: With limited data samples, the covariance matrix can become **ill-conditioned**, a state that amplifies estimation errors and leads to extreme, unstable weight allocations.
- **Hierarchical Risk Parity (HRP) Model**: HRP is a **machine learning algorithm** that uses **hierarchical clustering**. Developed as a robust alternative, HRP bypasses the need for matrix inversion by using this clustering method to identify the correlation structure between assets. It then allocates risk recursively to achieve a more balanced and stable portfolio.

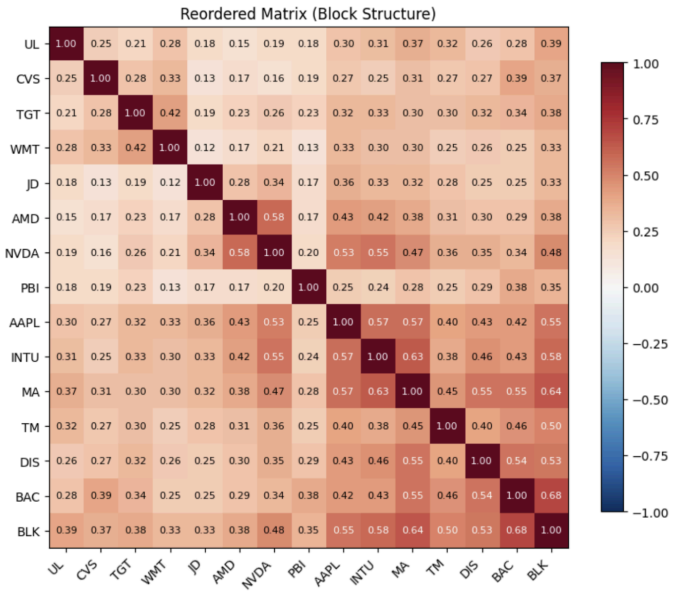
HRP Method: Correlation Clustering

The core of the HRP model is to identify the correlation structure of assets through hierarchical clustering, thereby building a more robust portfolio and avoiding the need for covariance matrix inversion.

Original Matrix: Displays the unprocessed correlation matrix of assets. The correlations are dispersed and lack a clear structure.



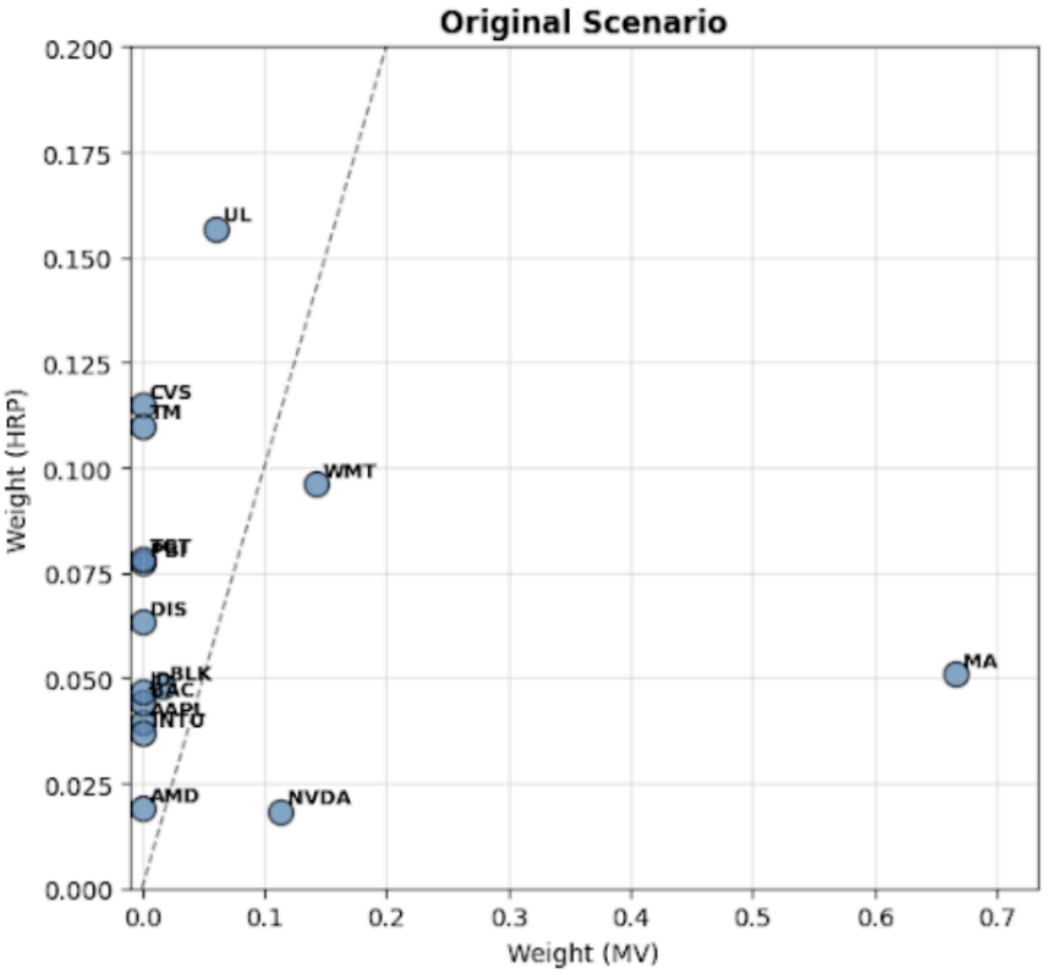
Clustered Matrix: Shows the reordered correlation matrix after applying the HRP algorithm, where highly correlated assets are grouped together to form distinct "blocks."



Weight Allocation: Concentration vs. Diversification

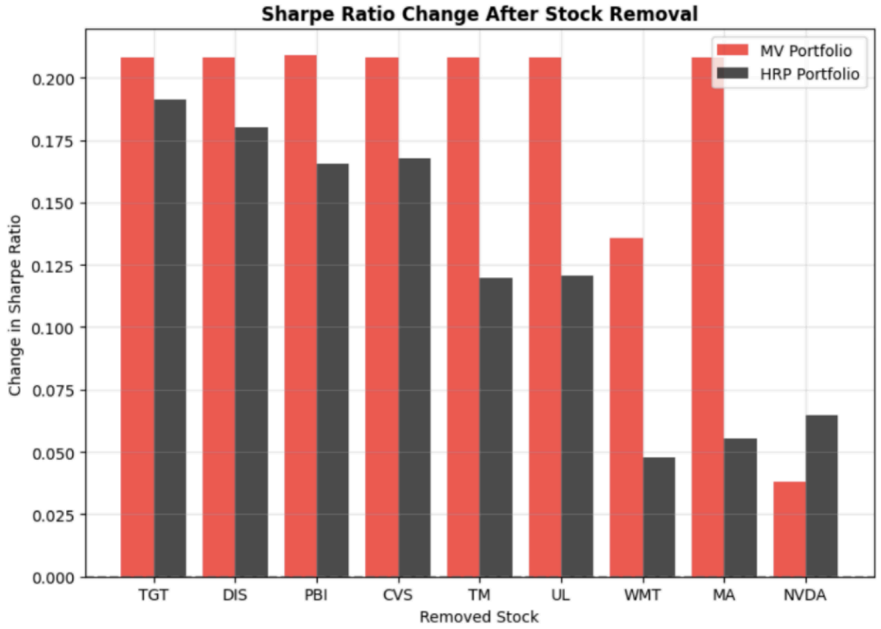
The fundamental difference between the MV and HRP models lies in their weight allocation strategies. The MV model tends to concentrate on a few assets, while the HRP model is dedicated to a broader diversification of risk.

- **MV's Concentration**: A few stocks, such as `MA`, receive extremely high weights in the MV model (points are on the right side of the chart), which reflects the model's concentrated allocation in pursuit of maximizing returns.
- **HRP's Breadth**: Stocks like `UL`, `CVS`, and `TM` receive higher weights in the HRP model (points are in the upper part of the chart), indicating that HRP spreads risk across a broader range of assets.
- **Key Insight**: HRP's allocation strategy allows for a wider diversification of risk, whereas MV may incur higher concentration risk in the pursuit of high returns.

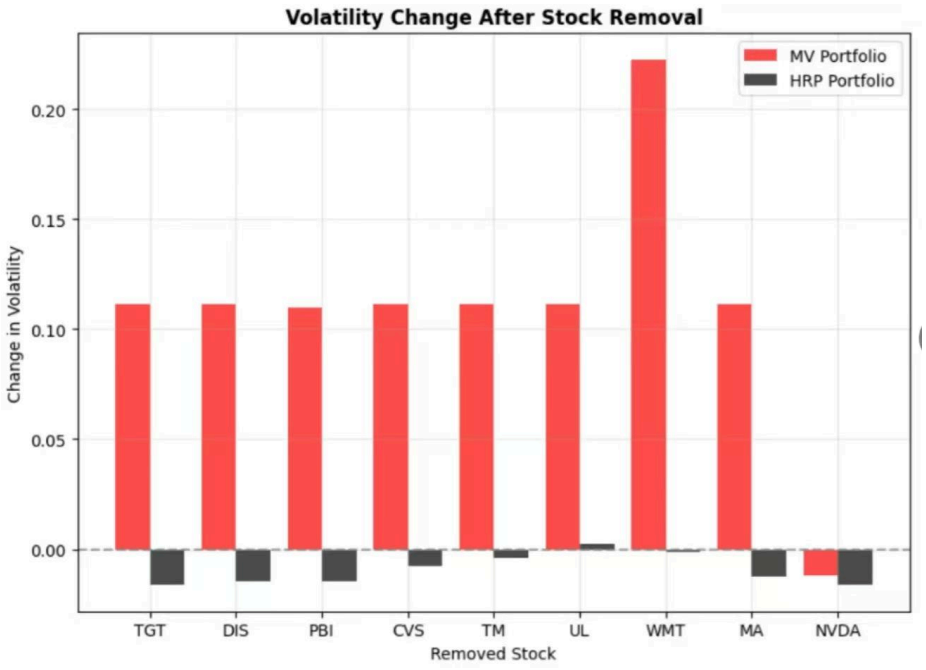


Robustness Analysis: Marginal Impact

- **Sharpe Ratio Change:** The removal of stocks like `TGT` and `DIS` significantly increased the Sharpe ratio for both models, suggesting these stocks may have been a drag on the original portfolio's risk-adjusted returns.



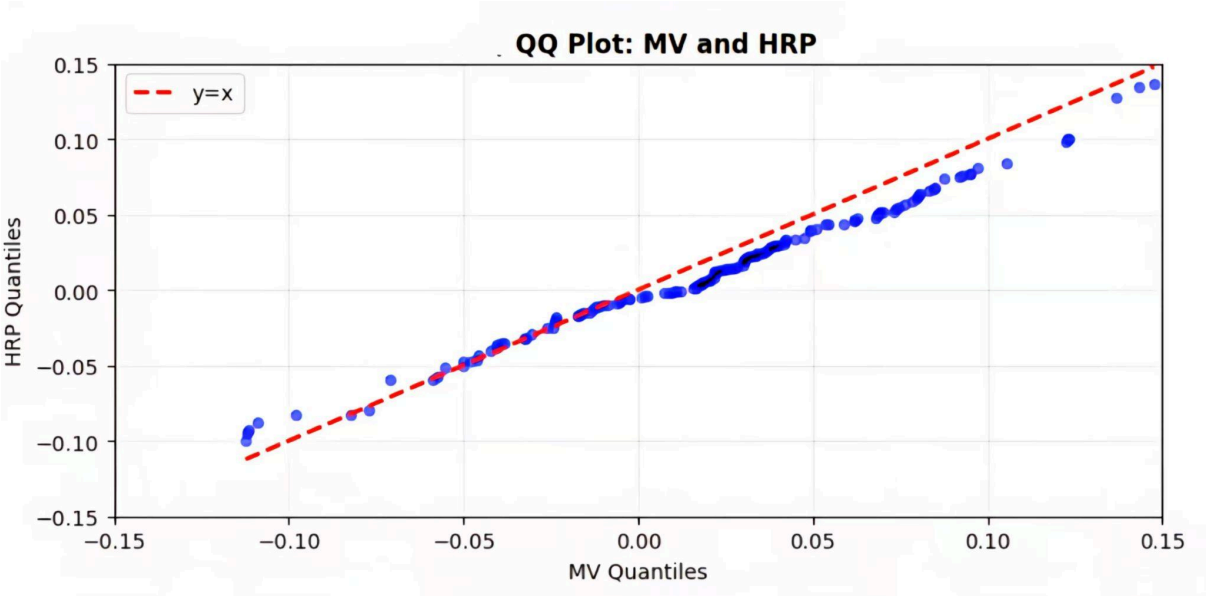
- **Volatility Change:** When `WMT` was removed, the MV portfolio's volatility rose dramatically, while the HRP portfolio's volatility remained largely unchanged. This demonstrates HRP's robustness.



Return Distribution Comparison: MV vs HRP

The Q-Q plot provides a direct visual comparison of the monthly return distributions for the MV and HRP models, revealing their performance differences in both typical and extreme market scenarios.

- **Central Performance:** In most return scenarios, the two models' performance is highly similar, with the data points clustering tightly around the baseline.
- **Tail Performance:** In extreme return scenarios, HRP's losses are smaller than MV's (less downside risk), but its gains are also smaller than MV's (limited upside potential).
- **Key Insight:** HRP's risk parity strategy effectively limits downside risk, but at the cost of sacrificing some upside gains.



	Skew	Excess Kurtosis	Jarque–Bera P Value
MV	-0.1957	0.0419	0.730
HRP	0.0140	0.2576	0.800

Core Conclusion and Trade-off

The empirical analysis in this study reveals a key trade-off between the Markowitz (MV) and Hierarchical Risk Parity (HRP) models: **Efficiency** versus **Robustness**.

- **MV's Efficiency**: In a favorable data environment (e.g., with a low covariance matrix *condition number*: the ratio of the largest to smallest eigenvalues), the MV model can effectively leverage its theoretical advantages to achieve higher average returns and superior downside risk-adjusted returns (a higher Sortino ratio).
- **HRP's Robustness**: Although the HRP model is more conservative in terms of returns, its risk diversification strategy, achieved through hierarchical clustering, makes it more **robust** to changes in asset composition. Its return distribution is closer to normal, and it is better at limiting extreme losses.

The Core Trade-off: Choosing the MV model means pursuing higher return efficiency but accepting higher concentration risk and greater sensitivity to data uncertainty. Conversely, choosing the HRP model means accepting more moderate returns in exchange for greater robustness and more predictable risk characteristics. The ultimate choice depends on the investment manager's core objective: whether to pursue efficiency or stability.

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