Machine Learning in Finance Workshop 2021

An Optimal Control Strategy for Execution of Large Stock Orders Using LSTMs

Brian Healy

Stanford University

Columbia University Data Science Institute

Columbia | Engineering
The Fu Foundation School of Engineering and Applied Science

Bloomberg

Overview

We consider the problem of executing large stock orders in a limit-order book using:

- Transaction costs modelled as a function of overall trading volume and convex in trade size.
- A long short term memory (LSTM) neural network to solve an optimization to minimize the total transaction costs accumulated when executing a large order.
- Industry-standard metrics such as time-weighted average price (TWAP) and volume-weighted average price (VWAP) to evaluate the strategy.
- The example of liquidating a large position in individual stocks over the course of a trading day.

We find, using recent market data, that that an LSTM strategy can out-perform TWAP and VWAP-based strategies when the size of the trade is very large.

The Problem

- Institutional investors will lose substantial amounts of money to /transaction costs/slippage if their trades are not carefully placed.
- When an order representing a significant proportion of the average daily volume of a stock needs to be executed, a naive strategy is to place a single, very large order on the exchange.
- This single order will consume most of the liquidity available in the limit-order book and will result in an average price per share that is far from the best bid/offer.

Our Proposal

- This liquidity issue can be avoided by sub-dividing the trade into smaller sub-orders, which are then executed at scheduled points over a fixed time period.
- We cast the sub-dividing of the trade into small sub-orders as an optimal control problem.
- We utilize a long short term memory (LSTM) neural network to solve the minimization of the total transaction costs through finding the optimal control strategy for sub-dividing the trade.

Background

- Liquidity is made available by market makers who submit limit orders at the different levels in an order book.
- Liquidity is consumed when a trader submits a market order to buy or sell.
- In the most basic case, a market order gets filled with limit orders available at the best available ticks. If the order is very large, then it will consume all limit orders at multiple ticks.

Background

- Market buy (sell) orders that consume multiple ticks will have a price per share equal to the best offer (bid) plus (minus) a transaction cost.
- Statistical studies of order books have shown that this transaction cost is roughly described as a convex function of the number of shares Almgren et al. (2005); Weber and Rosenow (2005) and that a better average price can be achieved by dividing the order into sub-orders to be executed sequentially over time. Almgren et al. (2005)
- The liquidity consumed by a market order refills quickly in the time between scheduled sub orders, and therefore a simple description of the problem need only to consider the temporary impact on price.
- An optimal sub-order policy will minimize an objective that considers the expected value of transaction costs over the entire trade.

Price Impact

- S_t^i is the best bid/offer of the stock $i = 1, 2, 3, \ldots, d$ at time t.
- V_t^i is the volume traded in the stock for $i = 1, 2, 3, \ldots, d$ at time t.
- The order book has limit-order distribution $\rho^i(t, s) \geq 0$, where the units of s are ticks relative to S_t^i.

An order of a^i shares consumes liquidity up to a relative price r_t^i such that

$$a^i = \int_1^{r_t^i} \rho^i(t, s) ds \tag{1}$$

Price Impact

Assuming a trade does not shock the market but still has price impact, (trade size 1% - 10% of total volume), a simple form for ρ is:

$$\rho^i(t,s) = \frac{V_t^i}{\epsilon}|s-1|^\beta \tag{2}$$

- ρ follows a power law with limit-orders distributed proportionally to volume
- $\beta \in [0,\infty)$ and $\epsilon > 0$ is a scaling parameter.
- Volume is typically quoted as the result of some form of moving-average calculation. Since we seek to optimize execution of a large order during a single trading day, V_t^i will be the volume of trades in the minute before time t.

Price Impact

When relative price r_t^i in (1) is computed with distribution ρ^i in (2), we see a price that is a concave function of order size divided by volume,

$$r_t^i = 1 + \text{sign}(a^i) \left(\frac{\epsilon(\beta+1)}{V_t^i} |a^i| \right)^{\frac{1}{\beta+1}} \tag{3}$$

- r_t^i can be viewed as the *impacted price*.
  - Almgren and Chriss (2001) had $\beta = 0$ so that the order book has equal liquidity at all ticks (impacted price is linear in a).
  - Many empirical studies conclude that impacted price is a concave function Almgren et al. (2005); Bershova and Rakhlin (2013); Bouchaud (2010); Cont et al. (2014), such as $\beta = 1$ (square-root function).

Price Impact

Combining (1), (2) and (3), the dollar amount of trading loss due to the price impact is:

$$\begin{aligned} \text{loss}(t,a) &= \sum_{i=1}^{d} S_t^i \left(\int_1^{r_t^i} s\rho^i(t,s)ds - a^i \right) \\ &= C_{\epsilon,\beta} \sum_{i=1}^{d} S_t^i (V_t^i)^{-\frac{1}{\beta+1}} |a^i|^{\frac{\beta+2}{\beta+1}} \end{aligned} \tag{4}$$

where $C_{\epsilon,\beta} = \frac{1}{\beta+2}(\epsilon(\beta+1))^{\frac{\beta+2}{\beta+1}}$.

- The convexity of (4) with respect to order size $|a^i|$ shows that large orders should be divided into sub-orders to reduce transaction costs.

Execution Policies

- Let X_t be a d-dimensional vector denoting the inventory in each of the stocks at time t (X_t^i is the number of shares of stock i at time t).
- Let the initial inventory be the d-dimensional vector X_0.
- Consider the situation where this inventory needs to be completely liquidated by terminal time T.
- An execution policy is a sequence of sub-orders, denoted by a_t for $t = 1, 2, 3, \ldots, T$ such that

$$X_t = X_{t-1} + a_t \ . \tag{5}$$

- That is, a_t^i is this execution policy's time-t sub-order for trading in stock i, and these sub-orders are chosen so that

$$X_T = 0 \ . \tag{6}$$

Execution Policies

Using the loss function given in (4), the total price for sub-order a_t^i is:

$$|a_t^i| S_t^i \left(1 + C_{\epsilon,\beta} \text{sign}(a_t^i) \left(|a_t^i| / V_t^i\right)^{1/(\beta+1)}\right) \tag{7}$$

and the average price is:

$$S_t^i \left(1 + C_{\epsilon,\beta} \text{sign}(a_t^i) \left(|a_t^i| / V_t^i\right)^{1/(\beta+1)}\right) \tag{8}$$

Time-Weighted Average Price (TWAP)

- Let $(a_t)_{t \leq T}$ be an execution policy.
- The TWAP for this policy is:

$$\overline{S}^i_T = \frac{1}{T} \sum_{t=1}^{T} S^i_t \left(1 + C_{\epsilon,\beta} \text{sign}(a^i_t) \left(\frac{|a^i_t|}{V^i_t} \right)^{\frac{1}{\beta+1}} \right) \tag{9}$$

- Commonly divide the order into pre-determined sub-orders $a^i_t = -X^i_0/T$ with sub-orders being submitted every n minutes.
- This **TWAP strategy** clearly guarantees that terminal condition (6) is met.
- It ignores the evolution of volume and other features of the trading period.

Volume-Weighted Average Price (VWAP)

- Let $(a_t)_{t \leq T}$ be an execution policy.
- The VWAP for this policy is:

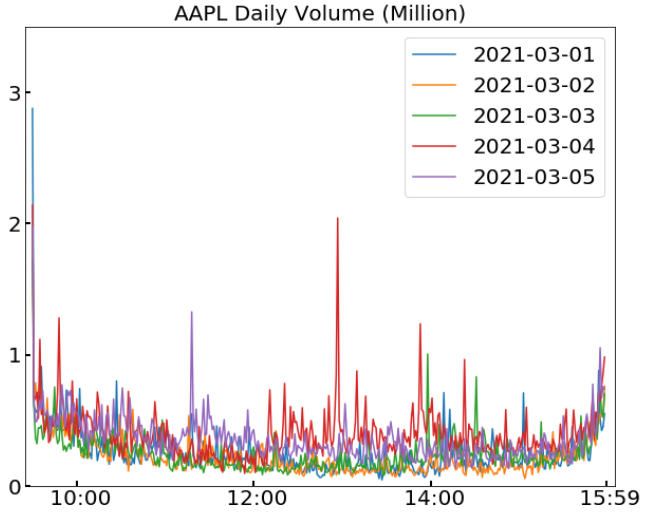
$$\overline{S}_V^i = \frac{1}{|x_0^i|} \sum_{t=1}^{T} |a_t^i| S_t^i \left(1 + C_{\epsilon,\beta} \text{sign}(a_t^i) \left(\frac{|a_t^i|}{V_t^i} \right)^{\frac{1}{\beta+1}} \right) . \tag{10}$$

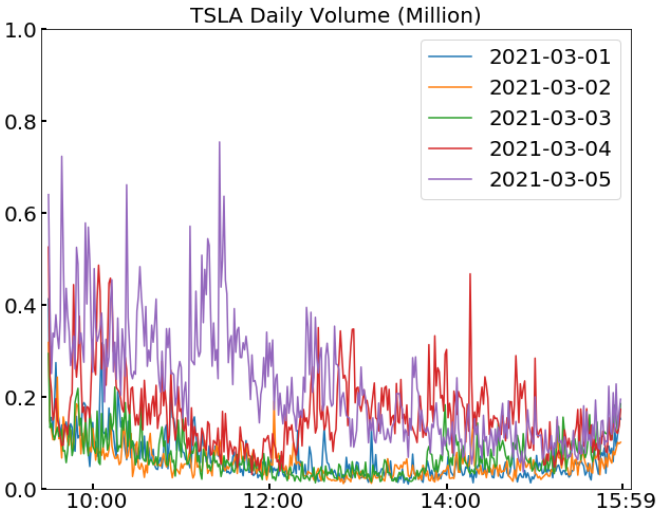
- A VWAP strategy requires knowledge of the typical evolution of volume over a trading period.
- We can compute averages $\overline{V}_t^i$ of the volume of stock i at time t and use VWAP strategy:

$$a_t^i = -\frac{X_0^i \overline{V}_t^i}{\sum_{t=1}^{T} \overline{V}_t^i} \qquad \text{for } t \leq T . \tag{11}$$

- This simple VWAP strategy can be effective for single-day execution because volume follows a predictable "U" shape.
- A VWAP strategy such as (11) does not guarantee that the VWAP of (10) is achieved, but it can be a close proxy if the estimated averages $\overline{V}_t^i$ have low noise.

Daily Volume





Process

- Let (S_t, V_t) be a Markov process
- We find the execution policy that minimizes the total trading losses solving solving:

$$\min_a \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} S_t^i (V_t^i)^{-\frac{1}{\beta+1}} |a^i|^{\frac{\beta+2}{\beta+1}}\right] \tag{12}$$
$$X_t = X_{t-1} + a_t \quad \text{and} \quad X_T = 0$$

- This optimization can be solved by training an LSTM network.
- The solution can be compared with a TWAP and VWAP strategy.
- We use 5-minute execution intervals for all strategies to ensure a fair comparison.

LSTM Design

- Algorithm 1 shows how the LSTM network is trained.
- We use the loss function given in equation (12).
- The output of the network is an inventory amount and an updated state for the LSTM network (the latter we denote as h).
- Time increments are 5 minutes but the intermediate data is still seen by the LSTM, which means the strategy is making full use of the available information.

Algorithm 1: Training of LSTM Architecture for Optimal Execution Every 5 Minutes In A 390-Minute Trading Day

```
Initialize parameters of LSTM units;
for k = 1 to NUM_EPOCH do
    X_0 = x_0, L = 0, t = 1;
    while t < 390 do
        X_t, h_t = LSTM(h_{t-1}, S_t, V_t);
        if mod(t,5)=0 then
            L += loss(t, X_t - X_{t-5});
        end
    end
    L += loss(390, -X_385);
    -Compute gradient of loss (backpropagation);
    -Update parameters (gradient descent step);
end
```

LSTM Design

- The flow of state variables through the LSTM network is shown in Figure 2 below.
- The network takes as input:
  - Current stock prices
  - Current volumes
  - Current time value
  - Current inventories
- The trading day has 390 minutes, and so the first trade occurs at time $t = 4$, with successive trades occurring every 5 minutes thereafter and the final trade occurring at minute 390 when the market closes.

LSTM Design

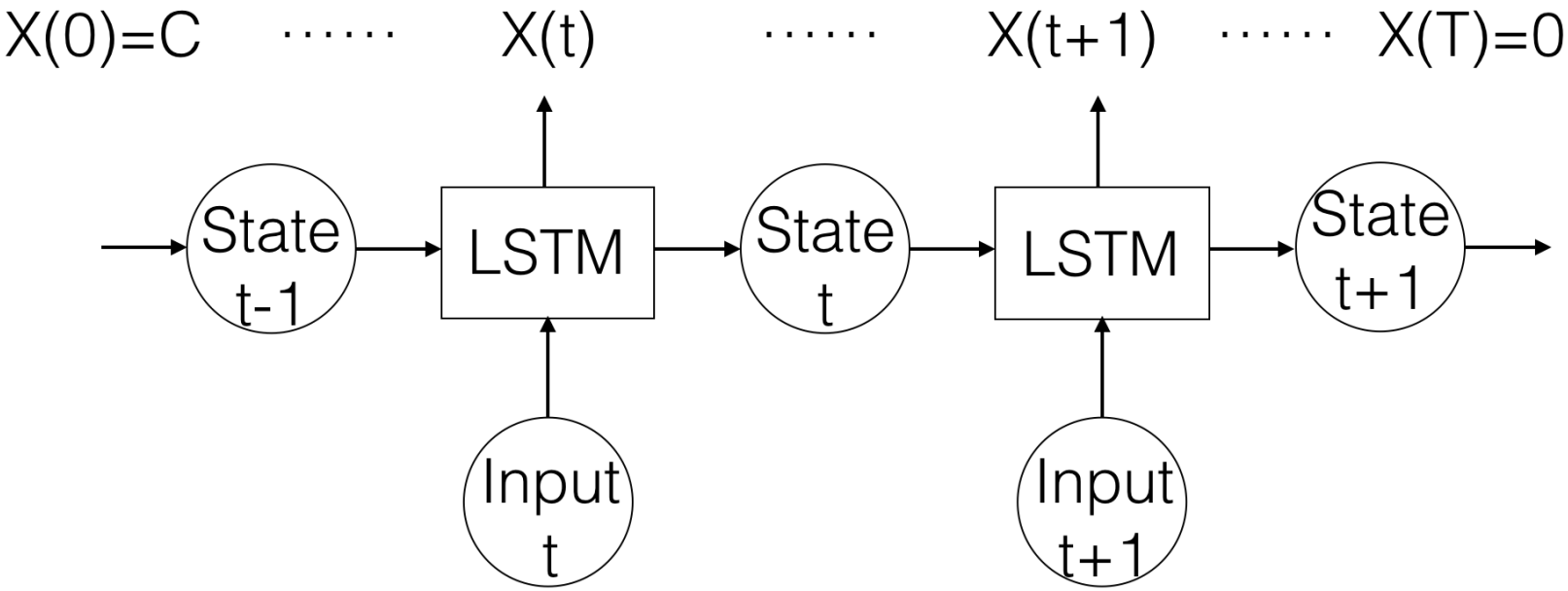


Figure 2: Flow of variables through the LSTM

Data

- We used minute-by-minute prices and average volumes from 7th December 2020 through 5th March 2021
- The 12 large capitalization stocks listed below.
- We chose firms from the tech sector, the financial sector and some healthcare companies.

Name	Avg Daily Vol (3m).	Shares Outstanding
Amazon	3.66M	503M
Facebook	19.9M	2.4B
Netflix	8.59M	434.85M
Google	1.76M	689.63M
Apple	111M	16.79B
Microsoft	29.3M	7.56B
Nvidia	8.1M	602.62M
Tesla	35M	1.08B
J.P. Morgan	14.8M	3.05B
Pfizer	35M	5.745B
Moderna	14.8M	400.5M
J & J	7.9M	2.633 B

LSTM Initialisation

We trained the LSTM network for the inventory liquidation problem with:

- $X_0^i = 500,000$ shares for each i.
- 2 hidden layers, each with 50 nodes.
- Sigmoid activation functions.
- 10,000 epochs.
- 51 days from December 7th 2020 to February 19th 2021.

LSTM Setup

Additional parameters:

- $\beta = .67$ so that the price impact in has a power of .6 as suggested by Almgren et al. (2005)
- $\epsilon = 10^{-4}$

These values imply (roughly) that demand is met by market makers who place orders within 35 basis points of the best bid/offer.

Timing:

- Using GPUs, the runtime for training was less than 20 minutes.
- Using CPUs, the runtime is around 10 hours.

Trading Costs for the 3 strategies

Date	LSTM	TWAP	VWAP
2/22	438,238.00	506,620.25	444,790.06
2/23	380,255.50	437,266.25	384,744.12
2/24	489,168.19	527,133.88	485,574.22
2/25	357,856.78	371,715.75	363,918.47
2/26	379,620.81	426,681.12	387,767.78
3/1	493,586.19	586,260.75	494,846.47
3/2	520,365.53	621,989.44	538,451.56
3/3	372,745.38	409,027.16	385,030.28
3/4	296,482.75	299,098.81	296,558.34
3/5	288,553.75	316,294.66	290,124.97
Mean	**401,687.31**	**450,208.81**	**407,180.59**

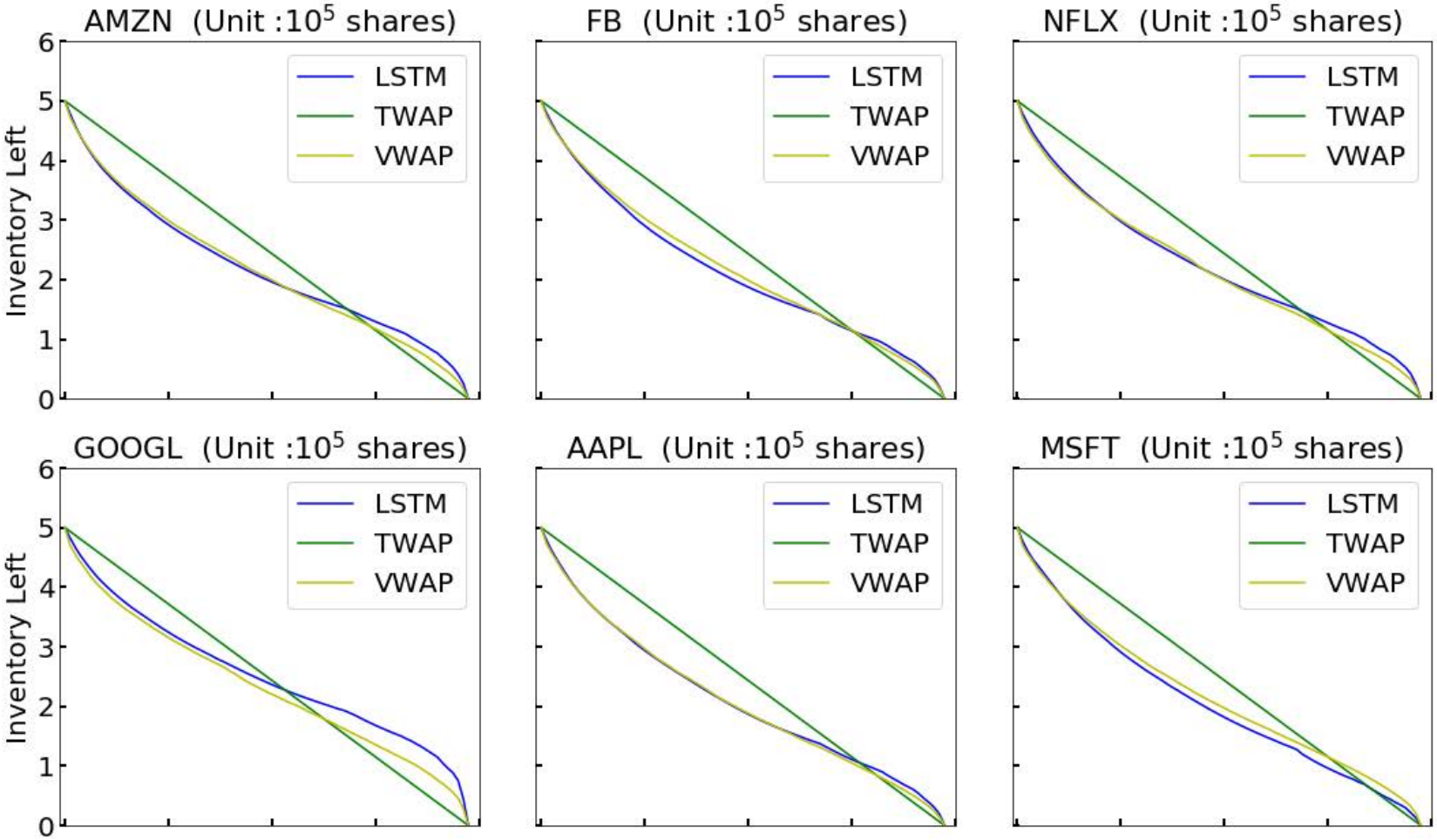
- In 9 out of 10 days of out-of-sample testing, the VWAP accumulated a loss greater than the LSTM strategy, with an overall mean loss that was almost $6,000 greater.

- The total running cost for each of the 3 strategies on day March 2, 2021.
- Each strategy executed the liquidation of 500,000 shares of each of the 12 stocks.
- The total loss computation is that given in (4) with $\beta = .67$.

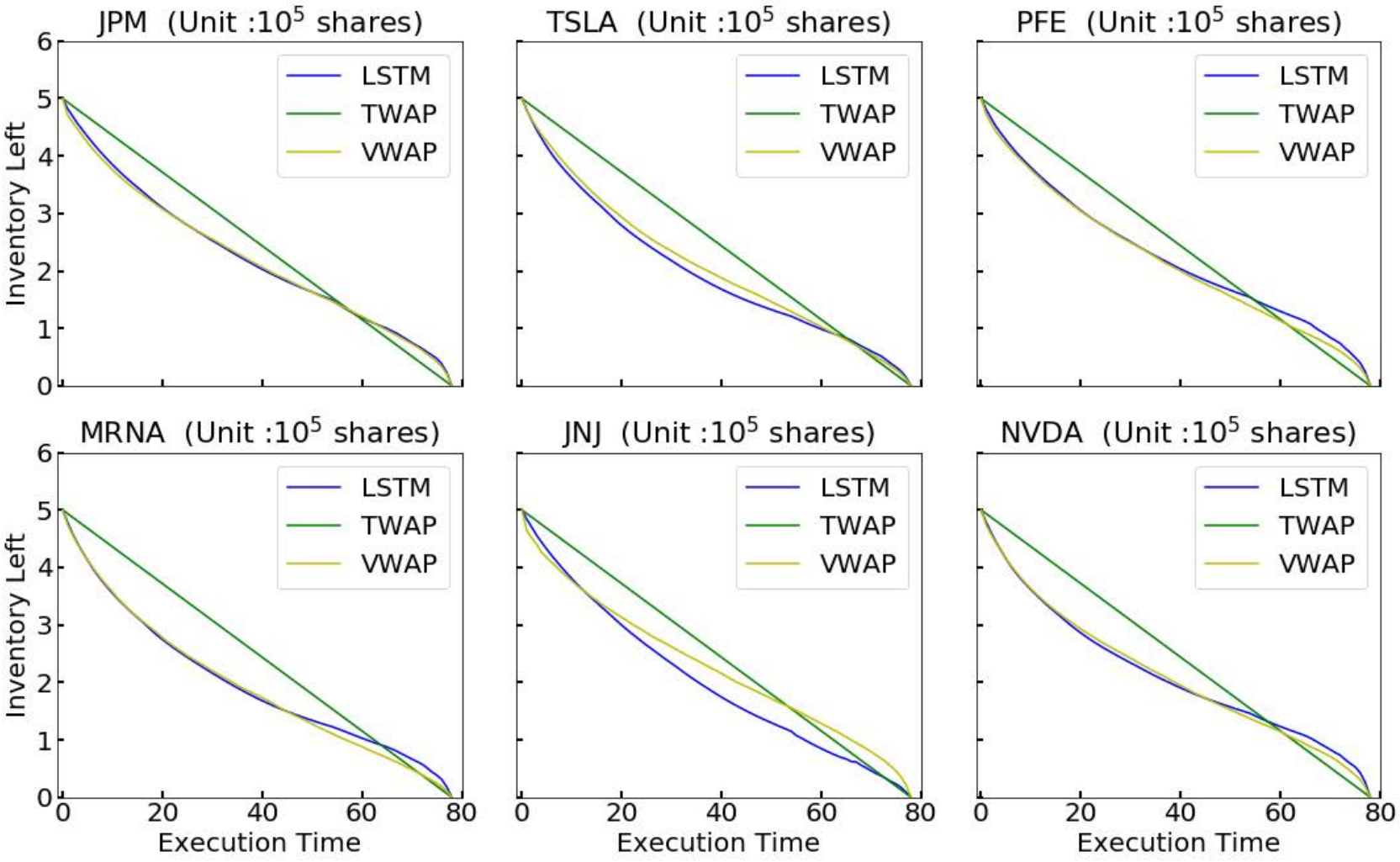
Inventory Process

- The general pattern for inventory liquidation for LSTM and VWAP can be described as an "S" shape relative to the linear TWAP
- High volume in the morning encourages the LSTM to submit larger sub-orders. VWAP behaves similarly was designed to.
- Similarly there is an increase in sub-order size in the final hour of the trading day, which is a time when volume is typically high.
- In particular, LSTM generates an increase in sub-order size in the final minutes of trading as it has learned that there is typically a surge in volume then.
- To avoid the costs associated with a failure to execute the entire inventory, the LSTM finds a non-myopic control sequence that will execute both in a timely way and with consideration given to the possibility of an end-of-day volume surge.

Inventory Process



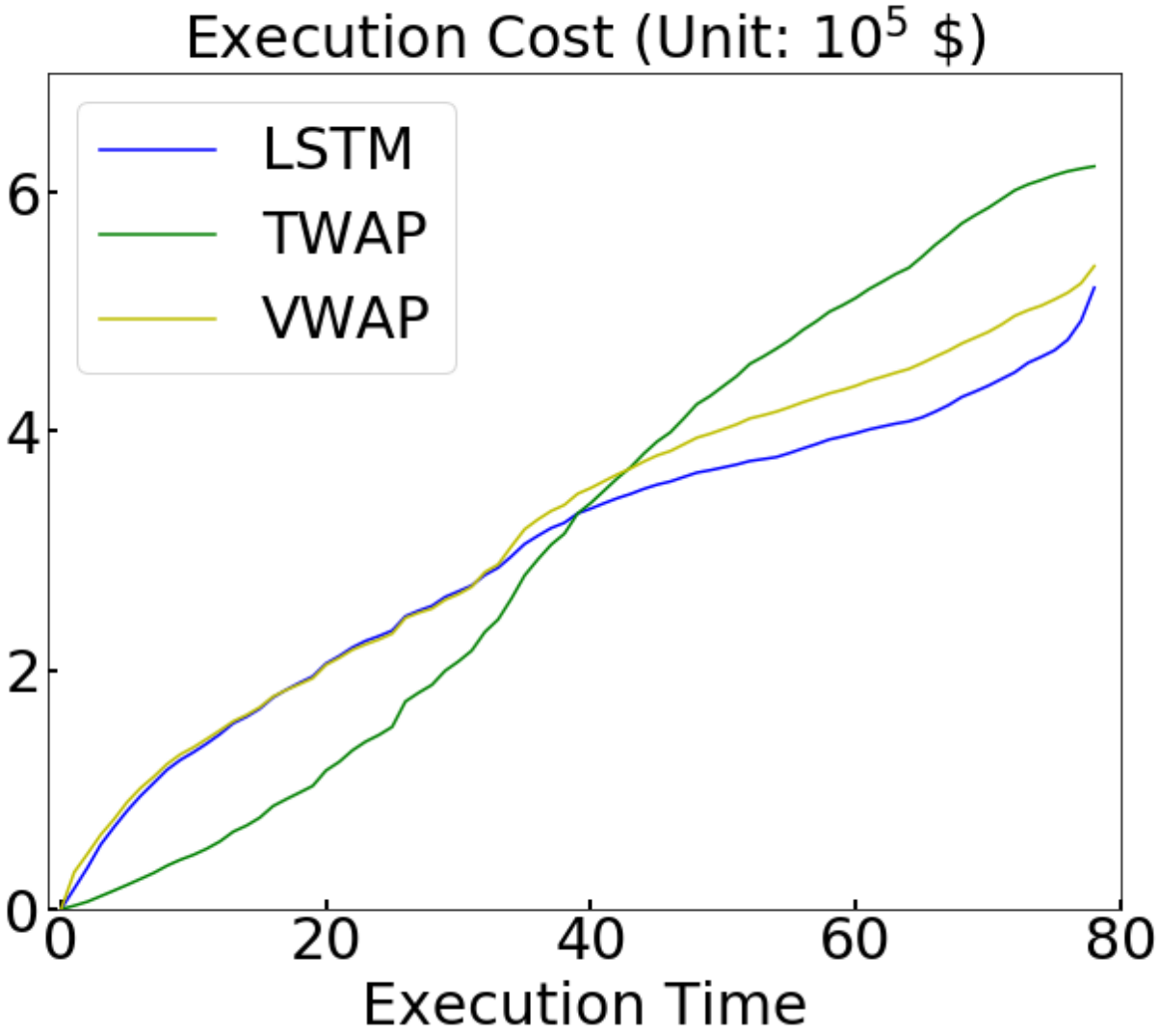
Inventory Process



Loss Process

- A typical day's pattern for total running loss also shows "S" shape for all strategies.
- As the TWAP doesn't execute as fast as either LSTM or VWAP in the morning it has lower losses.
- As the day progresses, the TWAP will incur higher transaction costs as it continues submitting orders during the mid-day hours when the volume is lower and the price action is not as favorable.
- We can see the acceleration in loss accrual by LSTM in the final hour of the day due to it's heavier volume in the closing munutes.

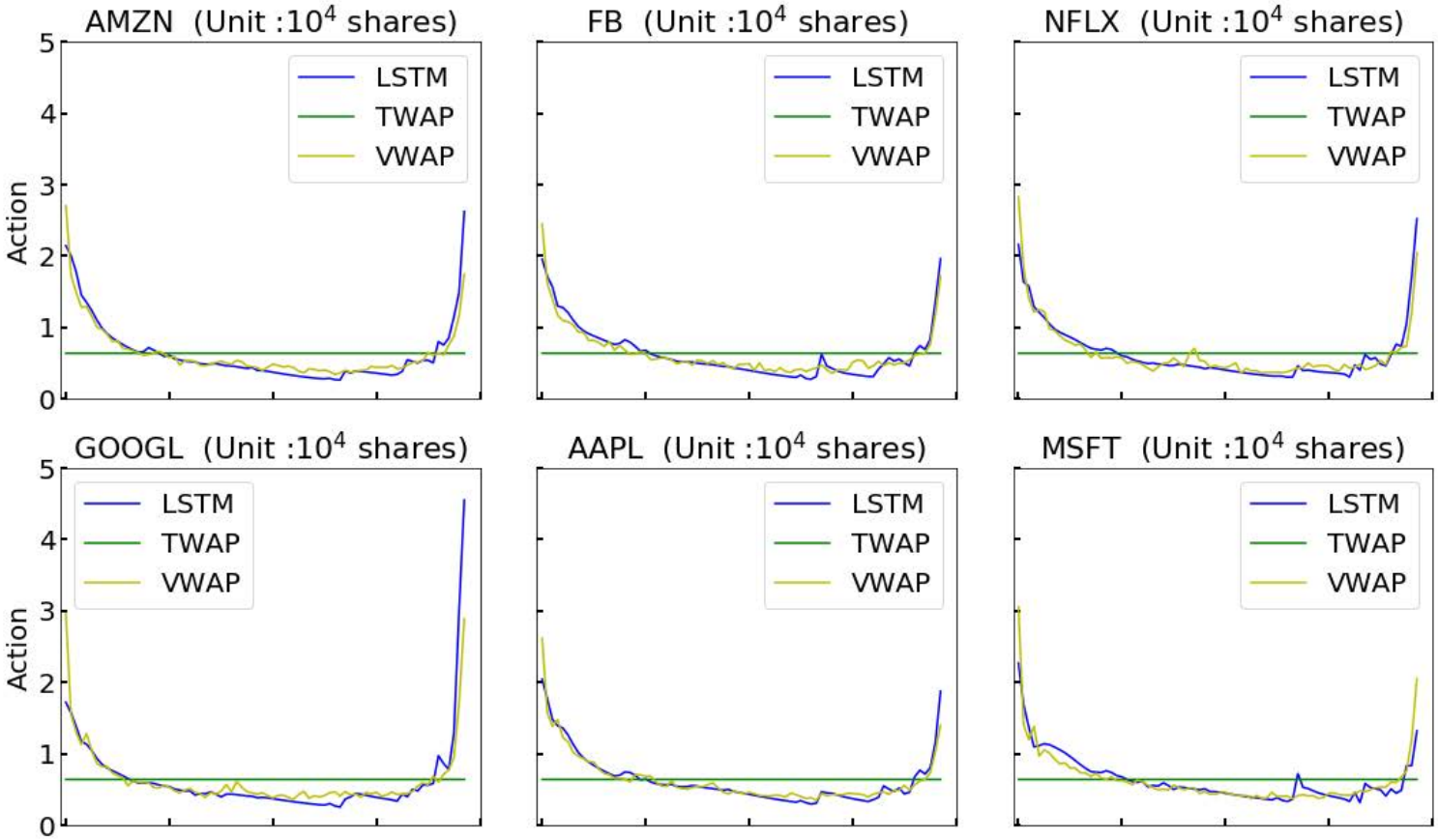
Loss Process



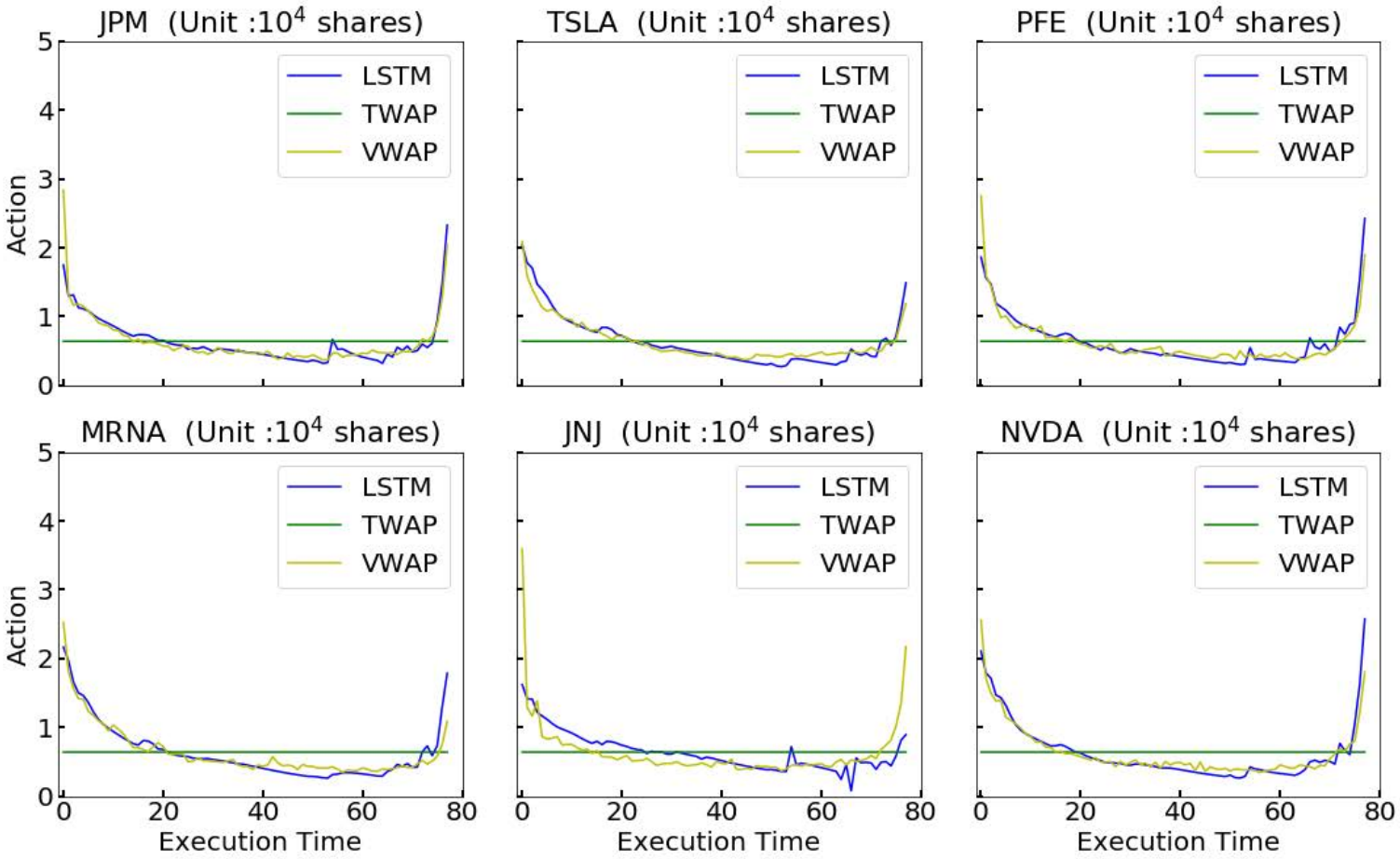
Actions

- Focusing on the actions taken, we see something similar to the inventory results.
- LSTM and VWAP submit large sub-orders at those times that typically have the highest trading volume.
- This is is by construction in the case of VWAP but has been learned by the LSTM from the training data.

Actions


AMZN (Unit :10^4 shares)
FB (Unit :10^4 shares)
NFLX (Unit :10^4 shares)
GOOGL (Unit :10^4 shares)
AAPL (Unit :10^4 shares)
MSFT (Unit :10^4 shares)
LSTM
TWAP
VWAP
Action
0
1
2
3
4
5

Actions


JPM (Unit :10^4 shares)
TSLA (Unit :10^4 shares)
PFE (Unit :10^4 shares)
MRNA (Unit :10^4 shares)
JNJ (Unit :10^4 shares)
NVDA (Unit :10^4 shares)
LSTM
TWAP
VWAP
Action
Execution Time
0
1
2
3
4
5
0
20
40
60
80

Comparison of VWAPs

Ticker	LSTM	TWAP	VWAP
AMZN	**3134.5156**	3134.4297	**3134.5034**
FB	263.8812	263.8801	263.8812
NFLX	550.8960	550.8882	550.8948
GOOGL	**2074.3865**	2074.2847	**2074.3630**
AAPL	126.1221	126.1219	126.1221
MSFT	234.8160	234.8151	234.8161
NVDA	545.7809	545.7782	545.7806
TSLA	702.4022	702.4010	702.4024
JPM	150.7749	150.7744	150.7751
PFE	33.6946	33.6945	33.6946
MRNA	149.4903	149.4900	149.4902
JNJ	159.4356	159.4355	159.4364

VWAP for the 3 execution strategies - 2nd March 2021.

Comparison of VWAPs

- The LSTM and VWAP strategies completed the liquidation with better average price than TWAP.
- LSTM does not consistently outperform the VWAP strategy.
- However, LSTM does outperform in the cases of Amazon and Google, both of which have average daily volume less than 2 million shares, which means that the 500,000 shares being executed are double-digit percentages of the daily volume.
- This suggests that LSTM outperforms when the trade size is extremely large, otherwise a simple VWAP strategy is fine.

Conclusion

- LSTM can be used for optimal execution of large stock orders in a limit-order book.
- The strategy clearly out-performs TWAP and is slightly better than VWAP-based strategies in out-of-sample testing.
- LSTM and VWAP produce similar results because volume is the primary factor for both.
- LSTM learns the role of volume in determining minute-by-minute transaction costs and manages to uncover some essential principles for large-order execution. It is possible that out-performance of the LSTM is due to its ability to aggregate across multiple stocks and detect effects such as heteroskedasticity among both the price and the volume time series, but future studies should be performed to verify these claims.

Questions ?

References

Robert Almgren and Neil Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.

Robert Almgren, Chee Thum, Emmanuel Hauptmann, and Hong Li. Direct estimation of equity market impact. *Risk*, 18(7):58–62, 2005.

Nataliya Bershova and Dmitry Rakhlin. The non-linear market impact of large trades: Evidence from buy-side order flow. *Quantitative finance*, 13(11):1759–1778, 2013.

Jean-Philippe Bouchaud. Price impact. *Encyclopedia of quantitative finance*, 2010.

Rama Cont, Arseniy Kukanov, and Sasha Stoikov. The price impact of order book events. *Journal of financial econometrics*, 12(1):47–88, 2014.

Philipp Weber and Bernd Rosenow. Order book approach to price impact. *Quantitative Finance*, 5(4):357–364, 2005.