# Empirical Lipschitz Variational TVD Estimator and Applications to Quantifying Distribution Shift

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## Introduction

- We build on the family of Integral Probability Metrics (IPM) and design a new distance metric between probability distributions that belong to this family. The new metric is termed Lipschitz Variational Total Variation Distance (LV-TVD) and is a relaxation of the integral probability metric representation of the well-known Total Variation Distance (TVD).
- We propose a procedure based on a linear program (LP) to consistently estimate this distance based on two empirical samples alone These estimates can provide meaningful and tight lower bounds for the TVD between two probability distributions. Applications include quantifying distribution shift and estimating the Neyman Pearson region.

## Lipschitz Variational TVD and Two-sample Estimator

Consider the following function class $\mathcal{F}_{LVD}^{l} = \{f: ||f||_{\infty} \leq 1, ||f||_{L} \leq l\}$, which can be used to define an Integral Probability Metric (termed LV-TVD) on two probability distributions $P, Q$ with densities $p, q$ on metric space $(\mathcal{S}, \rho)$:

$$\gamma_{LVD}^{l}(P,Q) = \sup_{f \in \mathcal{F}_{LVD}^{l}} \left| \int_{\mathcal{S}} f\, dP - \int_{\mathcal{S}} f\, dQ \right|$$

This represents a one-parameter family of IPMs where $l > 0$ is a Lipschitz smoothness parameter that controls the Lipschitzness of the chosen function class. This is related to the well-known TVD distance:

$$TVD(P,Q) = \int_{x \in \mathcal{S}} |p(x) - q(x)|\, dx$$

through inequality: $\forall l > 0, \gamma_{LVD}^{l}(P,Q) \leq TVD(P,Q)$.

LV-TVD is a metric on the space of probability distributions and it metrizes weak convergence like the Dudley metric. Larger $l$ results in larger LV-TVD.

With $\rho$ denoting the distance metric on data points, the following LP problem solves for the empirical LV-TVD based on two empirical data samples $P_m, Q_n$ of $P, Q$ and converges to the true LV-TVD as sample sizes increase. Here $\{X_i\}_{i=1}^{N}$ are combined from the two data samples ($N = m + n$). $\tilde{Y}_i = \frac{1}{m}$ for samples from $P_m$ and $\tilde{Y}_i = -\frac{1}{n}$ for samples from $Q_n$.

$$\begin{aligned} \gamma_{LVD}^{l}(P_m, Q_n) &= \max_{a_1,\ldots,a_N} \sum_{i=1}^{N} \tilde{Y}_i\, a_i \\ s.t. \quad & -l\rho(X_i, X_j) \leq a_i - a_j \leq l\rho(X_i, X_j), \forall i,j = 1,\ldots,N \\ & -1 \leq a_i \leq 1, \forall i = 1,\ldots,N \end{aligned}$$

## Numerical Demonstration on ETF Market Indices Data

We consider returns (from Yahoo finance) for six ETF indices from 09/2011 to 08/2021. The indices are iShares J.P. Morgan USD Emerging Markets Bond ETF (EMB), SPDR Bloomberg High Yield Bond ETF (JNK), iShares National Muni Bond ETF (MUB), Vanguard Developed Markets Index Fund (VEA), Vanguard 500 Index Fund (VOO), and iShares Russell 2000 ETF (IWM), representing the following markets respectively: emerging market debt, high yield bonds, municipal bonds, non-US equities, US large cap, and US small cap. For each index, there are 2500 trading days segmented into either five (non-overlapping) periods of 500 dates or ten periods of 250 dates to compute the average distribution shift score.

The distribution shift score for each period is defined by estimating the LV-TVD distance between each segment and the full data, as shown in Fig. 2 for two different window sizes. The final averaged distance estimate across all periods is the score reported in Table 1 for each index and each window size.

**Fig 1. Cumulative Returns of (Selected) Market Indices: US Large Cap (VOO) vs. Municipal Bonds (MUB)**

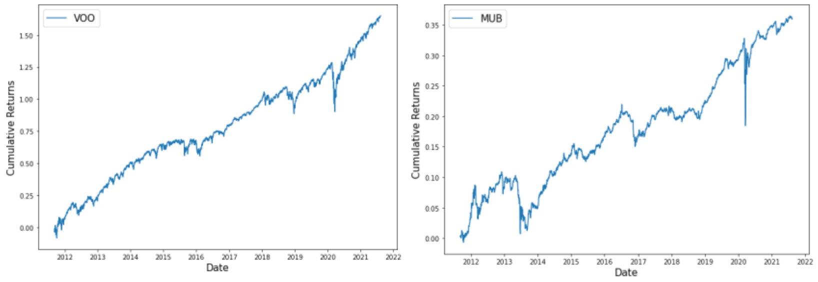


**Fig 2. Distribution Shift Scores of Market Indices for Different Window Sizes**

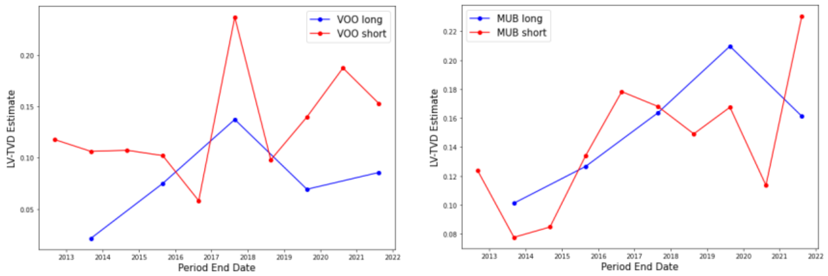


**Table 1. Average Estimated Distribution Shift Scores of Market Indices**

| Market ETF Index | Score (Window Size 500) | Score (Window Size 250) |
|---|---|---|
| US Large Cap (VOO) | 0.07762 | 0.13053 |
| Emerging Market Debt (EMB) | 0.09921 | 0.12729 |
| High Yield Bonds (JNK) | 0.11989 | 0.14372 |
| US Small Cap (IWM) | 0.12018 | 0.13800 |
| Non-US Equities (VEA) | 0.12527 | 0.14776 |
| Municipal Bonds (MUB) | 0.15250 | 0.14274 |

From these results, US Large Cap (VOO) can be seen as the most stable index with a small distribution shift score measured by either window size, and Municipal Bonds (MUB) can be seen as the most volatile, vice versa. The cumulative returns and distribution shift scores of these two indices are plotted in Fig. 1 and Fig. 2 respectively, providing intuitive verifications.

## Extensions and Future Directions

- The Lipschitz parameter in our proposed family of LV-TVD distance metrics can be adaptively chosen based on the data samples to achieve desirable properties such as scale invariance and obtain tighter lower bounds to the ground-truth TVD value.
- Extensions to generalized LV-TVD can be achieved by scaling two densities by respective constants. Its empirical estimator can be obtained similarly using an LP. These quantities are useful in describing the Neyman Pearson region.
- Distribution shift scores are used as an input in computing Explainability Index (EI) and Risk of Target (RoT), which are unifying risk metrics introduced by Hirsa et al. [4]. These metrics are applied to asset allocation, security selection, etc.
- Future directions of research include using the proposed estimators to perform change-point detection in streaming data.

## References

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