Approximate Risk Parity with Return Adjustment and Approximation Bounds

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Introduction

- Traditional risk parity provides a way of diversifying a portfolio while preventing excessive risk concentration, providing a way to construct portfolios good risk diversification. The volatility of risk parity portfolio lies somewhere between the minimum variance and the 1/n portfolio that has extensively been studied. However, risk parity portfolios or equal risk contribution portfolios (ERC) depend only on the covariance matrix of the asset universe and is agnostic to the returns of the assets. Efforts have been made to incorporate returns into risk parity and other risk budgeting methods, but these methodologies produce non-convex optimization problems that are difficult to solve or relaxations that have no guarantees on the portfolio obtained. This not only makes them hard to solve using numerical heuristics, but these heuristics give no guarantees on the risk diversification. In this paper, we adopt the principle of diversifying risk contributions to improve returns, by satisfying approximate risk parity whilst providing bounds on risk spread and taking returns into account. Mathematically, we provide algorithms (RAH, RAC, AERC), that bound the gap between the risk contributions or risk spread ($\mathcal{RS}$) and allows profitable assets to contribute more to a portfolio than would be allowed through regular risk parity.

We now present a table summarizing our algorithms. Before that we define risk spread as,

$$\mathcal{RS}^{\mathcal{A}}(\Sigma, r) = \max_{i,j}\{\mathcal{RC}_i(x) - \mathcal{RC}_j(x)\}$$

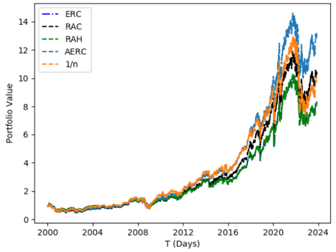
where, $\mathcal{RC}_i = \frac{(\Sigma x)_i x_i}{\sqrt{x^T\Sigma x}}$ is the risk of asset i

We also give a tight* naive bound,

$$\mathcal{RS}^{\mathcal{A}} \le \frac{\max_{i,j}\Sigma_{i,j} - \min\{0, \min_{i,j}\Sigma_{i,j}\}}{\sqrt{x^T\Sigma x}} \le \frac{\max_{i,j}\Sigma_{i,j} - \min\{0, \min_{i,j}\Sigma_{i,j}\}}{\mathcal{V}^*}$$

Here, $\mathcal{V}^*$ is optimal value of the minimum volatility problem,

$$\min_x \sqrt{x^T\Sigma x} \quad \text{s.t.} \sum_{i=1}^n x_i = 1$$



Algorithm	Sharpe Ratio*	MDD
AERC	3.65	0.261
RAC	3.54	0.211
RAH	3.42	0.224
1/n	3.21	0.408

Algorithms

Algorithm I: Risk Adjusted Holdings (RAH)

RAH adjusts the the risk contribution based on the return of each asset given by, $\frac{(\Sigma x)_i x_i}{\sqrt{x^T\Sigma x}}$

(where, $\mathbf{\Sigma}$ is the covariance matrix). We make the adjustment by introducing adjusted holdings-

$\tilde{x} \to x - \beta r$

Thus having the effect of decreasing the risk contribution by a rate that scales with the return of the asset. RAH then finds portfolio such that,

$\frac{(\Sigma x)_i \tilde{x}_i}{\sqrt{x^T\Sigma x}}$ is constant

Algorithm II: Risk Adjusted Contributions (RAC)

RAC allows risk contributions to scale linearly with return, i.e.

$\frac{(\Sigma x)_i x_i}{\sqrt{x^T\Sigma x}} = k r_i$

So simply put, we are allowing profitable assets to have a larger risk contribution.

Algorithm III: Approximately Equal Risk Contributions (AERC)

Here we solve an optimization problem, as shown in algorithm 4.
We show that the optimal solutions to this problem are nicely risk diversified w.r.t. total volatility for carefully

Algorithm 2 RAH (B)
Input: $\Sigma \succ 0 \quad r \ge 0$
Solve the problem

$$\text{argmin}\left\{-\sum_{i=1}^n \log((\Sigma x)_i) + \frac{1}{2}x^T\Sigma x + \beta(\Sigma r)^T x\right\}$$
$$\text{s.t.} \sum_{i=1}^n ((\Sigma x)_i x_i) \ge k \max_i (\Sigma x)_i$$
$$\Sigma x \ge 0$$

and obtain optimal solution x^*
return $\frac{x^*}{\sum_{i=1}^n x_i^*}$

Algorithm 3 RAC
Input: $\Sigma \succeq 0 \quad r \ge 0$
Solve the problem

$$\min_x \frac{1}{2}x^T\Sigma x - \beta\sum_{i=1}^n r_i \log x_i$$
$$\text{s.t.} \sum_{i=1}^n \log x_i \ge c$$
$$x \ge 0$$

and obtain optimal solution x^*
return x^*

Algorithm 4 AERC
Input: $\Sigma \succeq 0 \quad r \ge 0$
Solve the problem

$$\min_x \frac{1}{2}x^T\Sigma x - \beta\sum_{i=1}^n r_i \log x_i$$
$$\text{s.t.} \sum_{i=1}^n x_i = 1$$
$$0 \le x \le \mathbf{U}$$

and obtain optimal solution x^*
return $\frac{x^*}{\sum_{i=1}^n x_i^*}$

Summary of Algorithms with Proven Bounds

Algorithm	$\mathcal{RS}^{\mathcal{A}}(\Sigma, r) = \max_{i,j}\{\mathcal{RC}_i(x) - \mathcal{RC}_j(x)\}$	Output
RAH (IB)	$\frac{\max_{i,j}\Sigma_{i,j}\left(\frac{1}{n} + \frac{\beta \max_i r_i}{k}\right)}{\mathcal{V}^*}$	Increases (w.r.t ERC) the position on profitable assets
RAC	$\frac{\beta \max_i r_i}{\mathcal{V}^* n^2}$	Increases risk contribution of profitable assets, $\mathcal{RS}$ can be driven to 0 by tuning β, unlike RAH
AERC	$\frac{U\left(\frac{\sum_{i,j}\Sigma_{i,j}}{n^2} + \frac{\mathcal{V}^*}{k}\right) + \beta \max_i\{r_i\}}{\mathcal{V}^*}$	Mean variance type algorithm providing good bounds on $\mathcal{RS}$

Numerical Experiment: Same Σ, Increasing Returns

Here, we keep $\mathbf{\Sigma}$ the same across algorithms, and have a return $r_i = 0.2i$ for 10 assets. We now show the portfolio weights and the risk distribution of the different algorithms

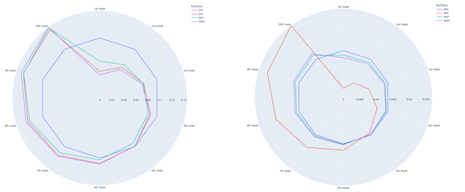


Figure 1: Left- Distribution of portfolio weights, Right- Distribution of risk contributions

Performance on Simulated GBM and Real World Data

We test our algorithms on simulated geometric brownian motion, mutual funds and on S&P 500 data. We present real world performance for all of our algorithms, along with plots for bounds for risk spread.

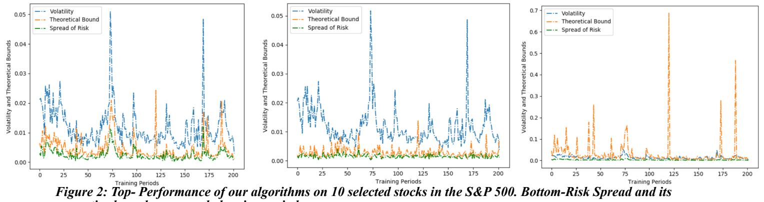


Figure 2: Top- Performance of our algorithms on 10 selected stocks in the S&P 500. Bottom-Risk Spread and its respective bounds across rebalancing periods

Asymptotic Properties of Risk Spread

We prove multiple asymptotic properties (in the number of assets) for the risk spread, the most important one being for AERC,

Lemma If $T_n := \frac{\sum_{i,j}\Sigma_{i,j}}{\mathcal{V}^*} = \frac{\sum_{i,j}\Sigma_{i,j}}{\sqrt{\sum_{i,j}\Sigma^{-1}_{i,j}}} = o(n^3)$, and we set $U = \frac{2}{n}$, then we have that for $n > 2$,

$$\lim_{n\to\infty} \mathcal{RS}^{\text{AERC}}(n) = 0$$

Here, $\Sigma \in \mathbb{R}^{n\times n}$, $\mathcal{V}^* = \text{argmin}_{x\in\mathbb{R}^n, e^T x = 1}\, x^T\Sigma x$

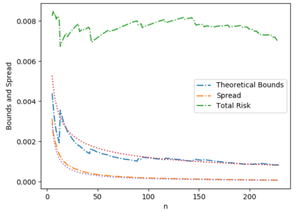


Figure 3: Asymptotic properties of risk spread

References

Scan code for complete list of references